Journal of Engineering Physics and Thermophysics, Vol. 82, No. 4, 2009

CALCULATION OF THERMAL REGIMES FOR HOMOGENEOUS AND COMPOUND BODIES OF ARBITRARY SHAPE

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UDC 536.24

We propose a new method for calculating the average (integral) temperature of homogeneous and compound bodies of arbitrary shape under cooling (heating) — a zonal integral method. Its chief advantages — simplicity, universality, and satisfactory exactness — have been demonstrated on known analytical solutions.

Keywords: *thermal regimes, method for calculating the average (integral) temperature, body of arbitrary shape.*

Introduction. The process of heat transfer in solid bodies in the absence of internal heat sources (sinks) is described by the differential heat conduction equation whose general form in Cartesian coordinates is

$$\rho c \frac{\partial t}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial t}{\partial z}\right). \tag{1}$$

The conditions at the interface S between a solid body and a labile medium are written in many practically important cases in the form of the Newton–Richman law

$$-\lambda \frac{\partial t}{\partial n}\bigg|_S = \alpha \left(t_{\text{env}} - t\,|_S\right). \tag{2}$$

In the case of a compound body, at the contact boundary of its parts Σ equality conditions of temperatures and heat flow on either size of this boundary (ideal contact conditions)

$$t\,|_{\Sigma_-} = t\,|_{\Sigma_+}, \quad -\lambda \frac{\partial t}{\partial n}\bigg|_{\Sigma_-} = -\lambda \frac{\partial t}{\partial n}\bigg|_{\Sigma_+} \tag{3}$$

are often taken. As initial conditions, the temperature distribution throughout the body

$$t\,|_{\tau=0} = t_0\,(x, y, z) \tag{4}$$

is given. As a result of the solution of this problem of nonstationary heat transfer (1)–(3), the temperature distribution in the bulk of any instant of time, i.e.,

$$t = t\,(\tau, x, y, z) \tag{5}$$

is determined.

In the general case (arbitrary region and initial distributions, temperature-dependent parameters, medium anisotropy), finding a solution of (5) is a complicated problem.

Analytical solutions of problem (1), (2), (4) have been found only for simple homogeneous bodies (plate, cylinder, and sphere) [1]. For compound bodies, such analytical solutions are known only in the limiting stationary case (e.g., steady heat transfer through a multilayer flat or cylindrical or spherical wall).

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An attempt to solve the cooling (heating*)) problem for bodies of arbitrary shape was realized in the theory of a regular thermal regime [2] in which the thermal process is broken down into two stages: the initial stage (irregular regime) and the stage of regular conditions. For the stage of the regular regime, "the relations between the rate of cooling, on the one hand, and the physical properties of the body, its shape, dimensions, and the cooling conditions, on the other hand", have been determined [3]. As for the irregular regime, the author of the theory [2] was faced with complicated mathematical problems (p. 147: "there is no need to speak of compound bodies — the case is apparently hopeless," p. 148: "even an approximate estimate of the duration of the irregular regime on the basis of the theory is impossible").

Thus, the theory of the regular thermal regime resolves the question on the direction of integral curves at the stabilization stage of the thermal regime but does not say how they fall on the required integral curve.

In many practically important cases, it is not at all necessary to know the temperature distribution inside the body (4), and it is enough to determine the average integral temperature $\bar{t}$ of the object of interest at any instant of time. The heat flow rate can in turn be determined from the average temperature [1].

Calculation of the Average Temperature of a Homogeneous Body. Consider the process of cooling of a body of arbitrary shape under the following assumptions simplifying the presentation of theoretical calculations of the proposed approach: t_{env} = const; α = const; the geometry of the body is invariable, i.e., V and S are constant; ρ and C are constant; inside the body neither phase transitions nor chemical reactions take place, i.e., phenomena followed by heat release or absorption are absent.

Suppose that both before and after each next time step the state of the body attains thermodynamic equilibrium, i.e., its temperature is equal to its certain average value. Integrating (1) over the entire volume V and over time from τ to $\tau + \Delta\tau$, and taking into account (2), we obtain

$$\rho C \Delta\bar{t} V = \int_{\tau}^{\tau+\Delta\tau} \left(\oint_S q_n dS \right) d\tau = \alpha \, (t_{\text{env}} - \bar{t}) \, S \Delta\tau$$

or

$$\Delta\bar{t} = \frac{\alpha}{\rho C} (t_{\text{env}} - \bar{t}) \frac{S}{V} \Delta\tau \, .$$

By virtue of the finite rate of heat exchange inside the body, this change in the average temperature $\Delta\bar{t}$ in time $\Delta\tau$ takes place not in the entire region of V but only in its part (zone) ΔV closest to the surface of contact with the environment. This is the point of the proposed method. The average temperature $\bar{t}$ at each subsequent instant of time $\tau + \Delta\tau$ is defined as

$$\bar{t}\big|_{\tau+\Delta\tau} = \frac{\bar{t}\big|_{\tau} V + \Delta\bar{t} \Delta V}{V} = \bar{t}\big|_{\tau} + \Delta\bar{t} \frac{\Delta V}{V} = \bar{t}\big|_{\tau} + \Delta\bar{t} \frac{S}{V} \frac{\Delta V}{S} \, . \tag{6}$$

The average size of the zone, namely its average thickness h, can be defined as $h = \Delta V/S$. To calculate it, let us make use of the condition of equality of the thermal flows at the interface between the two media (2). Admitting the hypothesis that the temperature change in a zone of thickness h is equal to the temperature difference between the body and the environment, we get

$$h = \frac{\lambda}{\alpha} \, . \tag{7}$$

As is seen, in this case the thickness of the zone of heat penetration through the surface of the body depends on the thermophysical property of the body (λ) and on the external conditions of heat exchange (α).

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*Hereinafter we will speak, for definiteness, of cooling, since heating differs from cooling only by the change of sign in the enthalpy change (body temperature).

Thus, from the known average temperature on the ith time layer $(\bar{t}_i)$ the average temperature of the body on the next $(i+1)$ time layer is found by the formula

$$\Delta\bar{t} = \frac{\alpha}{\rho C}(t_{\text{env}} - \bar{t}_i)\frac{S}{V}\Delta\tau\,, \quad h = \frac{\lambda}{\alpha}\,, \quad \bar{t}_{i+1} = \bar{t}_i + \Delta\bar{t}\frac{S}{V}h \tag{8}$$

or

$$\bar{t}_{i+1} = \bar{t}_i + \frac{\lambda}{\rho C}(t_{\text{env}} - \bar{t}_i)\left(\frac{S}{V}\right)^2\Delta\tau\,. \tag{9}$$

Interestingly, in (9) the average temperature does not depend on the external conditions of heat exchange. This is explained by the fact that these external conditions influence separately both the temperature change $(\Delta\bar{t})$ in the zone and the size of the zone itself (h). In so doing, as the heat exchange with the environment intensifies, $\Delta\bar{t}$ increases and the zone thickness h, conversely, decreases. In what follows, for brevity, we will call the given approach to the calculation of the average temperature of the body the zonal integral method (ZIM).

Analysis of the Proposed Method. Let us introduce the quantity $\bar{\vartheta} = (\bar{t} - t_0)/(t_{\text{env}} - t_0)$. Then (9) in dimensionless variables will be written as

$$\bar{\vartheta}_{i+1} = \bar{\vartheta}_i + (1 - \bar{\vartheta}_i)\frac{\lambda\Delta\tau}{\rho C}\left(\frac{S}{V}\right)^2 = \bar{\vartheta}_i + (1 - \bar{\vartheta}_i)\,\text{Fo}\,\frac{1}{i+1}\left(\frac{S}{V}\right)^2 l^2\,, \tag{10}$$

where $\text{Fo} = \lambda\tau/(\rho c l^2)$, $\bar{\vartheta}_0 = 0$.

From the similarity theory it is known that the solution of the boundary-value problem (1), (2), (4) depends on both the Fourier number and the Biot number. But the solution by the proposed method ZIM (9) does not depend on the Biot number. The field of application of the ZIM method by the Biot similarity number follows from the quite natural statement that the temperature perturbation zone at each time step should not cover the whole of the body, i.e., $\Delta V < V$. Hence $h = \Delta V/S < V/S$ or, in view of (7), $\lambda/\alpha < V/S$ or $\frac{1}{\text{Bi}} = \frac{\lambda}{\alpha l} < \frac{V}{S}\frac{1}{l}$. Consequently, the ZIM method is applicable for those problems for which the Biot number

$$\text{Bi} > \frac{S}{V}l\,. \tag{11}$$

Let us introduce auxiliary quantities $a = \frac{\lambda\Delta\tau}{\rho C}\left(\frac{S}{V}\right)^2$ and $b = 1 - a$. Then (10) will be given as

$$\bar{\vartheta}_{i+1} = \bar{\vartheta}_i + a(1 - \bar{\vartheta}_i) = a + b\bar{\vartheta}_i$$

or

$$\bar{\vartheta}_{i+1} = a(1 + b + b^2 + \ldots + b^i) + b^{i+1}\bar{\vartheta}_0 = a\frac{1 - b^{i+1}}{1 - b} = 1 - b^{i+1}\,.$$

At $\tau \to \infty$ $(i \to \infty)$ the body tends to thermodynamic equilibrium, i.e., $\bar{t} \to t_{\text{env}}$ or $\vartheta_{i+1} \to 1$. For this purpose it is essential that the second term in the last equality tends to zero. This is realized provided that $|b| < 1$. Hence we obtain a restriction on the time step

$$\Delta\tau < 2\frac{\rho C}{\lambda}\left(\frac{V}{S}\right)^2. \tag{12}$$

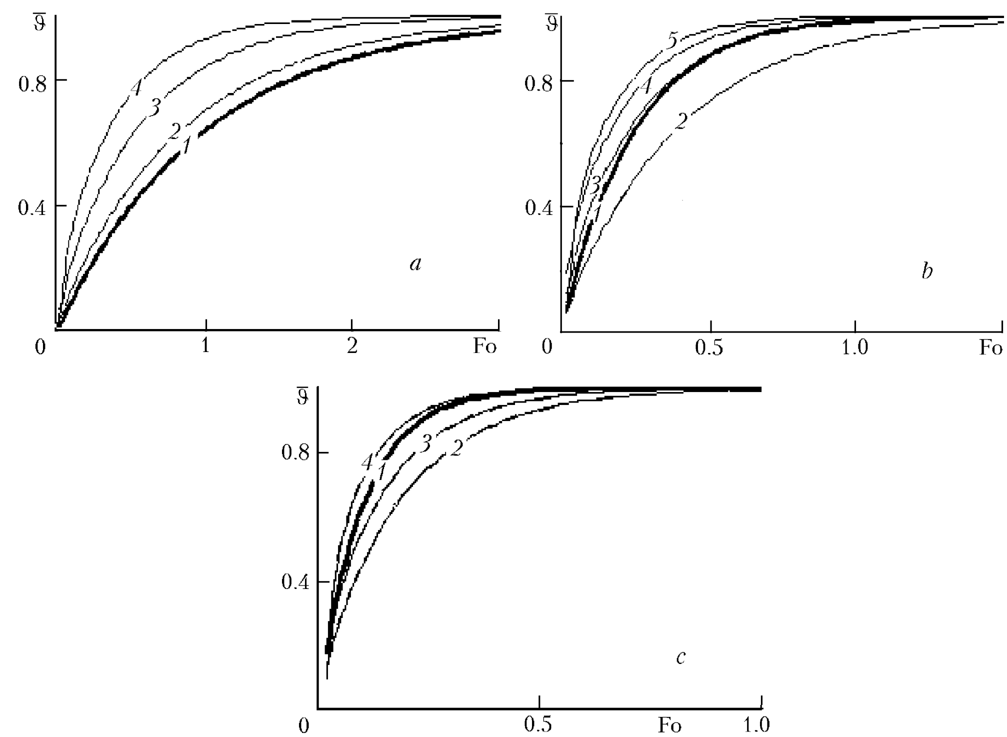

Fig. 1. Relative excess average temperature of the plate (a), the cylinder (b), and the sphere (c) as a function of the Fourier number: (a) 1) ZIM; 2) Bi = 2; 3) Bi = 5; 4) Bi = ∞; (b) 1) ZIM; 2) Bi = 2; 3) Bi = 5; 4) Bi = 11; 5) Bi = ∞; (c) 1) ZIM; 2) Bi = 3; 3) Bi = 5; 4) Bi = ∞.

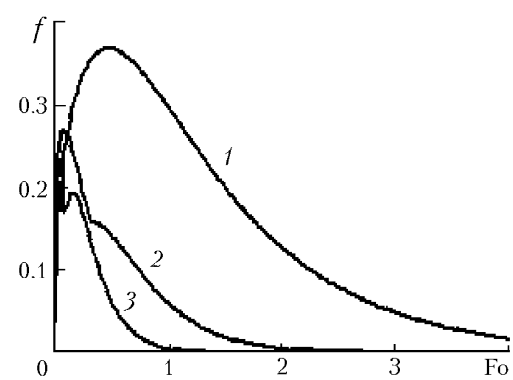

Fig. 2. Comparison of the analytical solutions with the numerical solutions by the ZIM method by the maximum deviation throughout the domain of definition of the problem: 1) plate; 2) cylinder; 3) sphere.

Evaluation of the ZIM Method. Let us demonstrate the potentialities of the ZIM method on the known analytical solutions of the heat transfer problems with boundary conditions of the third kind (1)–(3) [1], and, in so doing, the first six (although there are calculations with 18) terms were taken into account, which provides a very high accuracy of the solution.

Unbounded plate of thickness 2δ. In this case, the relation entering into (10) is $S/V = 1/\delta$ and $l = \delta$. The condition of (11) is $\text{Bi} > 1$. Figure 1a presents the dependence of the relative excess average (integral) temperature of the plate on the Fourier number for various Biot numbers (analytical solution) and the dependence obtained by the ZIM method.

Unbounded cylinder of radius R. Here $S/V = 2/R$, $l = R$, $\text{Bi} > 2$. The solution is given in Fig. 1b.

Sphere of radius R. In this problem, $S/V = 3/R$, $l = R$, $\mathrm{Bi} > 3$. The solution is given in Fig. 3c.

The maximum deviation of the numerical solution (ZIM) from the analytical solution in the absolute value f for all the three problems is given in Fig. 2. From the analysis of the figures it is seen that the greatest deviation of the analytical solution from the numerical one takes place for the problem with a plate, and this difference increases with increasing Biot number but does not exceed 37% also for $\mathrm{Bi} = \infty$. In the other problems (cylinder and sphere) this dependence is nonlinear for the Biot number. For the cylinder, the maximum difference does not exceed 28%, and for the sphere, 23%.

The accuracy of approximation of the analytical solution by the numerical one (by the ZIM method) increases sharply with increasing Fourier number, i.e., time. Figure 2 permits determining the instant of time from which the numerical solution will comply with the preset accuracy.

Calculation of the Average Temperature of a Compound Body. Let the body consist of two homogeneous parts. Denote by V_i (i = 1, 2) the volumes of the parts and by S the interface area. The whole of the body is heat-insulated from the environment. At the initial instant of time ($\tau = 0$) the temperature of the parts is equal, respectively, to t_{01} and t_{02}.

As in the case of a homogeneous body, integrating (1) over each of the volumes V_i, and over time, from τ to $\tau + \Delta\tau$, and taking into account (3), we get

$$\rho_1 C_1 \Delta\bar{t}_1 V_1 = \int_{\tau}^{\tau+\Delta\tau} \left(\oint_S q_n dS \right) d\tau = \lambda_1 \frac{\bar{t}_2 - \bar{t}_1}{h_1 + h_2} S\Delta\tau \,, \quad \rho_2 C_2 \Delta\bar{t}_2 V_2 = \int_{\tau}^{\tau+\Delta\tau} \left(\oint_S q_n dS \right) d\tau = \lambda_2 \frac{\bar{t}_1 - \bar{t}_2}{h_1 + h_2} S\Delta\tau \,,$$

where $h_1 + h_2$ is the total average thickness of the perturbed zone on either side of the contact surface. Hence we find the temperature change in the parts of the body in time $\Delta\tau$:

$$\Delta\bar{t}_1 = \frac{\lambda_1}{\rho_1 C_1} \frac{\bar{t}_2 - \bar{t}_1}{h_1 + h_2} \frac{S}{V_1} \Delta\tau \,, \quad \Delta\bar{t}_2 = \frac{\lambda_2}{\rho_2 C_2} \frac{\bar{t}_1 - \bar{t}_2}{h_1 + h_2} \frac{S}{V_2} \Delta\tau \,. \tag{13}$$

Likewise, we determine the average temperature $\bar{t}_i$ of each part of the compound body at each next instant of time $\tau + \Delta\tau$ as

$$\bar{t}_i \big|_{\tau+\Delta\tau} = \bar{t}_i \big|_{\tau} + \Delta\bar{t}_i \frac{\Delta V_i}{V_i} = \bar{t}_i \big|_{\tau} + \Delta\bar{t}_i \frac{S}{V_i} \frac{\Delta V_i}{S} = \bar{t}_i \big|_{\tau} + \Delta\bar{t}_i \frac{S}{V_i} h_i \,. \tag{14}$$

Proceed to the dimensionless variables by the formulas $\bar{\vartheta}_1 = (\bar{t}_1 - t_{01})/(t_{02} - t_{01})$ and $\bar{\vartheta}_2 = (t_{02} - \bar{t}_2)/(t_{02} - t_{01})$. Then (14) will take on the form

$$\bar{\vartheta}_{i+1,1} = \bar{\vartheta}_{i,1} + a\,(1 - \bar{\vartheta}_{i,1} - \bar{\vartheta}_{i,2}) \,, \quad \bar{\vartheta}_{i+1,2} = \bar{\vartheta}_{i,2} + b\,(1 - \bar{\vartheta}_{i,1} - \bar{\vartheta}_{i,2}) \,,$$

$$\bar{\vartheta}_{01} = 0 \,, \quad 0 \le \bar{\vartheta}_{i,1} < 1 \,, \quad \bar{\vartheta}_{02} = 0 \,, \quad 0 \le \bar{\vartheta}_{i,2} < 1 \,,$$

$$a = \frac{\lambda_1}{\rho_1 C_1} \left(\frac{S}{V_1} \right) \Delta\tau \frac{h_1}{h_1 + h_2} \,, \quad b = \frac{\lambda_2}{\rho_2 C_2} \left(\frac{S}{V_2} \right) \Delta\tau \frac{h_2}{h_1 + h_2} \,.$$

Substituting sequentially the known (calculated) temperature on the previous time layer into the formula for determining the temperature on the next layer, we obtain

$$\bar{\vartheta}_{i+1,1} = a\,(1 + c + c^2 + \ldots + c^i) = a \frac{1 - c^{i+1}}{1 - c} = \frac{a}{a + b} (1 - c^i) \,,$$

$$\bar{\vartheta}_{i+1,2} = b\,(1 + c + c^2 + \ldots + c^i) = b \frac{1 - c^{i+1}}{1 - c} = \frac{b}{a + b} (1 - c^i) \,, \tag{15}$$

$$c = 1 - a - b\,.$$

In the limit at $\tau \to \infty$ ($i \to \infty$), the temperatures of both parts of the body become equal ($\bar{t}_\infty$), which agrees (proceeding from the determination of the above-introduced dimensionless relative temperature) with the equality $\bar{\vartheta}_{\infty 1} + \bar{\vartheta}_{\infty 2} = 1$. From (15) it follows that $\bar{\vartheta}_{\infty 1} = a/(a+b)$ and $\bar{\vartheta}_{\infty 2} = b/(a+b)$, hence we arrive at the same equality. The necessary condition for such limiting convergence is $|c| < 1$ or $a + b < 2$. From this follows the restriction on the time step

$$\Delta\tau < 2\,\frac{h_1 + h_2}{\dfrac{\lambda_1}{\rho_1 C_1}\left(\dfrac{S}{V_1}\right)^2 h_1 + \dfrac{\lambda_2}{\rho_2 C_2}\left(\dfrac{S}{V_2}\right)^2 h_2}\,. \tag{16}$$

Since the enthalpy of the whole of the body has not changed, then

$$\rho_1 C_1 V_1\,(\bar{t}_\infty - t_{01}) = \rho_2 C_2 V_2\,(t_{02} - \bar{t}_\infty)$$

or (having divided both parts by $t_{02} - t_{01}$)

$$\rho_1 C_1 V_1\,\frac{a}{a+b} = \rho_2 C_2 V_2\,\frac{b}{a+b}\,.$$

Hence $\dfrac{a}{b} = \dfrac{\rho_2 C_2 V_2}{\rho_1 C_1 V_1}$. On the other hand, proceeding from the determination of the parameters a and b, we have $\dfrac{a}{b} = \dfrac{\lambda_1 h_1}{\rho_1 C_1 V_1^2}\,\dfrac{\rho_2 C_2 V_2^2}{\lambda_2 h_2}$. Comparing both equalities, we get

$$\frac{h_1}{h_2} = \frac{\lambda_2 V_1}{\lambda_1 V_2}\,. \tag{17}$$

In view of (17) the restriction on the time step (16) can be written in a more compact form

$$\Delta\tau < \frac{2}{\lambda_1 \lambda_2}\,\frac{\lambda_1 \dfrac{V_2}{S} + \lambda_2 \dfrac{V_1}{S}}{\dfrac{1}{\rho_1 C_1}\dfrac{S}{V_1} + \dfrac{1}{\rho_2 C_2}\dfrac{S}{V_2}}\,. \tag{18}$$

In view of (17) the formulas for calculating the average integral temperature of the body components by the ZIM method (14) on the next time layer will take on the final form

$$\bar{t}_{i+1,1} = \bar{t}_{i,1} + \frac{\lambda_1}{\rho_1 C_1}\,(\bar{t}_2 - \bar{t}_1)\left(\frac{S}{V_1}\right)^2 \Delta\tau\,\frac{\lambda_2 V_1}{\lambda_1 V_2 + \lambda_2 V_1}\,,$$

$$\bar{t}_{i+1,2} = \bar{t}_{i,2} + \frac{\lambda_2}{\rho_2 C_2}\,(\bar{t}_1 - \bar{t}_2)\left(\frac{S}{V_2}\right)^2 \Delta\tau\,\frac{\lambda_1 V_2}{\lambda_1 V_2 + \lambda_2 V_1}\,. \tag{19}$$

Heat Transfer through the Two-Layer Wall. This problem was chosen for evaluating the ZIM method due to the fact that: 1) its analytical solution in the stationary case is known and presented in all manuals on heat transfer (including [3, 4]); 2) both the compound body and the heat exchange with the environment are present simultaneously.

Knowing the temperature distribution across the two-layer wall at stationary heat transfer, let us determine the average integral temperatures of the layers:

$$\bar{t}_1 = t_{\text{env}1} - \left(\frac{1}{\alpha_1} + \frac{\delta_1}{2\lambda_1}\right)\frac{t_{\text{env}1} - t_{\text{env}2}}{R_0}, \quad \bar{t}_2 = t_{\text{env}1} - \left(\frac{1}{\alpha_1} + \frac{\delta_1}{\lambda_1} + \frac{\delta_2}{2\lambda_2}\right)\frac{t_{\text{env}1} - t_{\text{env}2}}{R_0}, \quad R_0 = \frac{1}{\alpha_1} + \frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2} + \frac{1}{\alpha_2},$$

where α_1, α_2 are the coefficients of heat transfer with the environment on different sides of the wall; δ_i, λ_i are the thickness and the heat transfer coefficient of the ith layer; $t_{\text{env}1}$ and $t_{\text{env}2}$ are the temperatures of the environment on different sides of the wall.

In dimensionless variables $\bar{\vartheta}_1 = (\bar{t}_1 - t_{\text{env}1})/(t_{\text{env}2} - t_{\text{env}1})$ and $\bar{\vartheta}_2 = (\bar{t}_2 - t_{\text{env}1})/(t_{\text{env}2} - t_{\text{env}1})$ this solution has the following form:

$$\bar{\vartheta}_1 = \frac{\frac{1}{\alpha_1} + \frac{\delta_1}{2\lambda_1}}{R_0} = \frac{1 + \frac{\text{Bi}_1}{2}}{1 + \text{Bi}_1\left[1 + \frac{\lambda_1\delta_2}{\lambda_2\delta_1}\left(1 + \frac{1}{\text{Bi}_2}\right)\right]}, \quad \bar{\vartheta}_2 = \frac{\frac{1}{\alpha_1} + \frac{\delta_1}{\lambda_1} + \frac{\delta_2}{2\lambda_2}}{R_0} = \frac{1 + \text{Bi}_1\left(1 + \frac{\lambda_1\delta_2}{2\lambda_2\delta_1}\right)}{1 + \text{Bi}_1\left[1 + \frac{\lambda_1\delta_2}{\lambda_2\delta_1}\left(1 + \frac{1}{\text{Bi}_2}\right)\right]}, \tag{20}$$

here, $\text{Bi}_1 = \alpha_1\delta_1/\lambda_1$ and $\text{Bi}_2 = \alpha_2\delta_2/\lambda_2$.

According to the proposed method ZIM, the time dependence of the average integral temperature of the wall layers is defined by the formulas

$$\bar{t}_{i+1,1} = \bar{t}_{i,1} + a_1(t_{\text{env}1} - \bar{t}_{i,1}) + b_1(\bar{t}_{i,2} - \bar{t}_{i,1}), \quad \bar{t}_{i+1,2} = \bar{t}_{i,2} + a_2(t_{\text{env}2} - \bar{t}_{i,2}) + b_2(\bar{t}_{i,1} - \bar{t}_{i,2}), \tag{21}$$

where

$$a_i = \frac{\lambda_i}{\rho_i C_i}\left(\frac{S_i}{V_i}\right)^2 \Delta\tau = \frac{\lambda_i}{\rho_i C_i \delta_i^2}\Delta\tau; \quad b_1 = \frac{\lambda_1}{\rho_1 C_1}\left(\frac{S}{V_1}\right)^2 \Delta\tau \frac{\lambda_2 V_1}{\lambda_1 V_2 + \lambda_2 V_1} = \frac{\lambda_1}{\rho_1 C_1 \delta_1}\Delta\tau \frac{\lambda_2}{\lambda_1\delta_2 + \lambda_2\delta_1};$$

$$b_2 = \frac{\lambda_2}{\rho_2 C_2}\left(\frac{S}{V_2}\right)^2 \Delta\tau \frac{\lambda_1 V_2}{\lambda_1 V_2 + \lambda_2 V_1} = \frac{\lambda_2}{\rho_2 C_2 \delta_2}\Delta\tau \frac{\lambda_1}{\lambda_1\delta_2 + \lambda_2\delta_1}.$$

Here the second terms in (21) correspond to the temperature change in the layer due to the heat exchange with the environment (9), the area of the surface of contact with the environment is S_i, and the ratio $V_i/S_i = \delta_i$. The third terms stand for the influence of the heat exchange between the layers (19) through the contact surface with area S, the ratio $V_i/S = \delta_i$.

For comparison with the analytical solution (20), we turn to the same dimensionless variables. Then we write (21) in the form

$$\bar{\vartheta}_{i+1,1} = \bar{\vartheta}_{i,1} - a_1\bar{\vartheta}_{i,1} + b_1(\bar{\vartheta}_{i,2} - \bar{\vartheta}_{i,1}), \quad \bar{\vartheta}_{i+1,2} = \bar{\vartheta}_{i,2} + a_2(1 - \bar{\vartheta}_{i,2}) - b_2(\bar{\vartheta}_{i,2} - \bar{\vartheta}_{i,1}), \quad 0 < \bar{\vartheta}_{i,1} < 1, \quad 0 < \bar{\vartheta}_{i,2} < 1.$$

In the limit at $\tau \to \infty$, a stationary regime of heat transfer sets in. The average temperatures of the layer are found from the solution of the following linear system:

$$\bar{\vartheta}_{\infty,1} = \bar{\vartheta}_{\infty,1} - a_1\bar{\vartheta}_{\infty,1} + b_1(\bar{\vartheta}_{\infty,2} - \bar{\vartheta}_{\infty,1}), \quad \bar{\vartheta}_{\infty,2} = \bar{\vartheta}_{\infty,2} + a_2(1 - \bar{\vartheta}_{\infty,2}) - b_2(\bar{\vartheta}_{\infty,2} - \bar{\vartheta}_{\infty,1}).$$

These relative temperatures are equal to

$$\bar{\vartheta}_{\infty,1} = \frac{a_2 b_1}{a_1 a_2 + a_2 b_1 + a_1 b_2} = \frac{\frac{\delta_1}{2\lambda_1}}{\frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2}}, \quad \bar{\vartheta}_{\infty,2} = \frac{a_1 a_2 + a_2 b_1}{a_1 a_2 + a_2 b_1 + a_1 b_2} = \frac{\frac{\delta_1}{\lambda_1} + \frac{\delta_2}{2\lambda_2}}{\frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2}}. \tag{22}$$

To determine the deviation of the numerical solution of the problem by the ZIM method (22) from the analytical solution (20), we analyzed the functions

$$f_1\left(\mathrm{Bi}_1, \mathrm{Bi}_2, \frac{\lambda_1\delta_2}{\lambda_2\delta_1}\right) = \overline{\vartheta}_1 - \overline{\vartheta}_{\infty,1}\,, \quad f_2\left(\mathrm{Bi}_1, \mathrm{Bi}_2, \frac{\lambda_1\delta_2}{\lambda_2\delta_1}\right) = \overline{\vartheta}_2 - \overline{\vartheta}_{\infty,2}\,.$$

The domain of definition of these functions, as follows from restrictions (11), is $\mathrm{Bi}_1 > 2$, $\mathrm{Bi}_2 > 2$, $\lambda_1\delta_2/\lambda_2\delta_1 > 0$. Dropping the calculation of the partial derivatives of both functions by all arguments and their analysis for sign constancy, we note that f_1 has a maximum value equal to 0.171, and its minimum value is equal to –0.05; the respective values of f_2 are 0.05 and –0.172 (the calculation accuracy was 0.001). This means that throughout the domain of definition of the initial problem the maximum deviation of the solution obtained by the ZIM method from the analytical one does not exceed 17%.

Conclusions. The proposed method ZIM (9), (19) for determining the average integral temperature of a body of arbitrary shape in different thermal regimes is simple and universal. With modern computational technologies, there is no need to speak of requirements in memory and count time — they are negligible. In its existing form, the ZIM method can be used as an engineering method. Its prospects are that it makes it possible to solve problems that require an optimal solution depending on either the average temperature or the heat flow rate. Seeking the extremum of the objective function in such problems is connected with a repeated solution of the most direct problem (this is hundreds, thousands of access). If the exactness of the obtained solution is not suitable, recommendations (approximate solution) will serve as the starting condition for a more thorough (but also more laborious) investigation.

This work was supported by the Russian Basic Research Foundation, grant No. 06-08-96916.

NOTATION

a, *b*, *c*, auxiliary quantities; $\mathrm{Bi} = \alpha\delta/\lambda$, Biot similarity number; *C*, specific heat capacity, J/(kg·K); Fo, Fourier similarity number; *f*, function of deviation of the analytical solution from the numerical one; *h*, average thickness of the perturbed zone, m; *l*, characteristic linear size of the problem, m; *n*, normal to the surface; *q*, heat flow density, W/m^2; R_0, thermal impedance of the heat transfer of the wall, m^2·K/W; *S*, surface area of the body, m^2; *t*, temperature, ^{o}C; $\overline{t}$, average integral temperature, ^{o}C; *V*, volume of the body, m^3; *x*, *y*, *z*, linear coordinates, m; α, coefficient of heat exchange with the wall, W/(m^2·K); $\Delta\overline{t}$, change in the average temperature of the body in time $\Delta\tau$, ^{o}C; ΔV, volume of the perturbed zone, m^3; $\overline{\vartheta}$, relative excess average temperature; λ, heat conductivity coefficient, W/(m·K); ρ, density, kg/m^3; τ, time, sec. Subscripts: env, environment; *i*, time layer number; 0, initial; 1, 2, part number of the compound body; ∞, parameters of the stationary regime.

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